THE

MATHEMATICAL GAZETTE

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Edited by E. M. LANGLEY, M.A.

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HERBART'S VIEW OF THE PLACE OF MATHEMATICS IN EDUCATION.

THE claims of Herbart on the attention of the teacher are at last being recognised in this country. Not merely is he deserving of study as one of the founders of modern psychology, not merely does he attract our interest as the central object of veneration of a rapidly increasing school in his native country and in the United States, but he also takes up a position apart from most philosophers, in that he actually devoted a large part of his life to the work of teaching, and deliberately applied his psychology to education. To the readers of this Gazette, the opinions of Herbart, who offers us this rare combination of the theorist of the first rank with the successful teacher of boys and men, will be of additional interest when it is realised that he plays somewhat the same rôle as the staunch advocate of Mathematics in the curriculum that Herbert Spencer does for Natural Science. He urges that "for six hours weekly" Mathematics should be "the beginning, middle, and end" of every system of instruction. And the fact that *Mathematik* is of wider connotation than we should now award to it will not deprive his remarks of much of their value. It will be as well, however, to bear in mind that he includes in this term all that is not comprised in *Geschichte.*

Let us first consider his monograph on Pestalozzi's *Idee eines ABC der Anschauung,* published ninety-two years ago, in which the principle of observation is developed and applied to the whole realm of education.

After drawing attention to the radical danger that besets the untrained sight-perception, owing, he argues, to its concentration on "colour," he maintains that it must be met by *Anschauung,* which, by fixing the attention on "form," will give the necessary correction. The senses can only discover what the mind of the child has been trained to see, so that the concept of "form" is one of the most important for the child to grasp. Now, Mathematics is the ideal subject for the cultivation of *Anschauung,* because it contains the substance of all that is necessary for the study of "form"; it is the best instrument for exhibiting to the child both the limit of his powers and his potential capacity. "In Mathematics, and nowhere else, is to be sought the thread for the child's early instruction, which can be so conditioned that it shall provide for its own use, as well as for all other studies, an authority at whose command distraction shall disappear and attention arise and persist." That is why "Mathematics is the beginning, middle, and end of every course of instruction."

But Mathematics is not to be taught for its own sake. The key-note to Herbart's position is to be found in the sentence:—"No one can be expected to think himself into the strict uniformity of Nature, who has had no training in the rigorous discipline of Mathematics and its deductions."

Let us note his standpoint. The studies included under the name of Humanities form a complete whole, with a central idea, viz. interest in Man; we must group the Natural Sciences into a complete whole with a central idea—interest in Nature. And with this, interest in Mathematics stands in close relation. Or, in other words, a many-sided interest, to use his own expression, is to be the source of apperception, and all the various divisions of the subject are to be taught for the sake of the light they throw on Nature. As the knowledge of the student grows, he is to study the natural sciences with reference to the light that Mathematics can throw on them. "The true place and rank of the natural sciences are not yet sufficiently determined; when these are defined, their inseparable companion, Mathematics, will also be given possession of its rights." And again, "From simple Arithmetic

to the Higher Mathematics, all must be connected with the study of Nature and with practical experiments in order to gain admission to the inner thought of the pupil." As soon as the most elementary Mathematical instruction forms for itself an isolated mass of concepts, it becomes "unpedagogic and at the same time it ceases to exert its full influence on the character." "To the real scholar a knowledge of Mathematics is indispensable, because without it a thorough knowledge of natural science is absolutely impossible."

In connection with Herbart's method, it is worth while examining his remarks on synthetic instruction. "The object of synthetic instruction is twofold; it must supply the elements and *prepare* their combination. The most general kind of synthesis is the *combinative*." This is to be exercised as soon and as often as possible. "It holds sway principally in the province of empiricism, where nothing hinders it from affording the recognition of the (logical) *possible*, of which the *incidental real* forms a part, and under which it may be classified in many ways. From this point it finds its way to the practical sciences, where it is the helper when successions of ideas have to be supplied to successive aspects of a given manifold. In the sphere of speculation the loss of this combinative synthesis may be sorely felt if it is wanting; this mathematicians have realised. Alike here, and in the sphere of taste, it becomes obscured through the peculiar kind of synthesis which rules therein, and which partly rejects inadmissible connections, and partly removes the mind from all irrelevant play of thought. Closely connected with the combinative concepts are those of *number*. Each combinative act is made up of a number of elements of complexion; number itself is the abstraction of these. . . . Speculative synthesis proper, entirely different from the logical-combinative, rests on relationships (Beziehungen). But the method of the relationships no one knows, and it is not the business of education to exhibit it. Neither is it the business of early years to take up a critical attitude towards nature. . . . The teacher must seek out, entirely regardless of *his* system, the least *dangerous* ways to prepare capacity for investigation as much as possible, and to awaken on many sides the impelling feeling aroused by single problems—the elements of speculation—for fear the young thinker should believe he will be soon at the end of his search. The safest without doubt is the study of Mathematics; unfortunately it has degenerated too much into a game assisted by lines and formulæ. Let it be led back as far as possible, to the thinking out of the *concepts* themselves." "Synthetic instruction supplies a number of new presentations, and has to work them out. It should observe constantly whether it *overfills* the mind or leaves it *too empty*. We shall find here, that not only the capacities, but also the disposition, varies at different times, and the treatment must be regulated accordingly. Further, government and discipline, but above all, the whole concentration of the teacher on the subject in hand, should operate in arousing an effort to comprehend everything completely, correctly, and immediately, and to grasp it clearly and luminously. Finally, beware of building *too quickly* on ground newly prepared. What is clear to-day is obscure again to-morrow; and he who is thinking laboriously on the particular cannot connect or apply. As to the elements, we must take care, whenever it is possible, that they lie ready long before they are wanted; further we ought always to build on a somewhat broad foundation, that there may be work to do, now here and now there, and change may thus be ensured. With respect to what is complex, it is very important, as far as possible, to occupy the mind with its forms, that it may anticipate the method and search for it itself. . . ."

Herbart holds that it is a mistake to allow the child to remain too long in a narrow circle; he must be always advancing, and at the same time learning to apply what he has learned. "If the object were simply to awaken self-activity, the rudiments would be quite enough to afford any number of problems, in which the pupils would enjoy the skill they had acquired, and would even feel delight in their own little discoveries, without ever learning the extent of the science." He then compares many problems to jests, legitimate and enjoyable at the proper time, but not to be allowed to interfere with the time for work. He thinks that the teacher should not stop to explain what will a little further on become evident. This reminds us of the remark made by Professor Chrystal in the preface to his Algebra:—"Every mathematical book that is worth anything must be read 'backwards and forwards,' if I may use the expression. I would modify Lagrange's advice a little and say, 'Go on, but often return to strengthen your faith.' When you come on a hard or dreary passage, pass it over; and come back to it after you have seen its importance or found the need for it further on."

But to resume. Problems dealing with the facts of Natural Science are infinitely more serviceable than mere practice problems, as the sciences are more readily explained by the use of mathematics when it stands in relation to technical knowledge. Again, the subject cannot be begun too early. That aptitude for this subject is more rare than for other studies is a fallacy, due, no doubt, to the late period at which the subject is begun. Further, mathematicians have neglected to place themselves on the same plane as their pupils. In arithmetic they have begun with simple combinations and geometrical forms, and have attempted to teach demonstrations to the child in whom the imaginative faculty is still lying dormant. Lines, angles, geometrical figures should be used in the teaching of number. "I have suggested marking out with bright nails on a board the typical triangles, and placing them continually within sight of the child in its cradle." Plane surfaces of varying forms, and angles, are to be carefully measured. The measurement of angles in degrees is the first preparation for geometry and trigonometry. Algebra should be introduced earlier than at present, in connection with number, under the form of literal Arithmetic, and soon assuming the form of general symbolical statement. "We ought to point out at a very early stage numerous examples of combinative operations, chiefly of the variations most commonly used. To this stage belong also the forms of space, at first squares and circles, as appearing oftenest in surrounding objects without analysis, and then angles. For illustration we use the hands of a clock, the opening of doors and windows, etc. Angles of 90°, 60°, 45°, 30° must first be pointed out. My *ABC der Anschauung*, which has its place here, presupposes a knowledge of this." And yet again: "The essential thing is the training of the eye in estimating distances and angles, and the connecting of this training with easy arithmetical problems. The object is not simply to improve the power of observing visible objects, but in particular to awaken the geometrical imagination, and to connect with it the arithmetical thought. Herein lies that necessary preparation for mathematics which is so often slighted. The means must be concrete objects. . . . The best to commence with are triangles cut from thin pieces of hard wood. Seventeen pairs of these are needed, all right-angled, and with one side of the same length in common. To find these triangles, draw a circle, radius 4 inches, and then draw tangents, secants, etc. for 5°, 10°, 15°, etc. to 85°. The different uses of these triangles may be readily imagined. The pupils themselves must actually measure the tangents and secants, and record the lengths correctly—at first only in integers and tenths. On these are based simple problems, the immediate object of which is that the pupil may become accustomed to the observation of simple objects. . . . When he begins to measure plane surfaces, the wooden triangles are laid aside, and the geometrical construction takes the place of actual perception of a concrete object. At the same time, Arithmetic and Algebra begin to treat of simple proportion, and later of powers, roots, logarithms." So in every department, the subject is applied directly to some natural object or phenomenon. Thus the perception is trained.

[An excellent translation of *Die Aesthetische Darstellung der Welt als das Hauptgeschaft der Erziehung* (1804), and the *Allgemeine Pädagogik aus dem Zweck der Erziehung abgeleitet* (1806), has been published by Swan Sonnenschein and Co. (1892), with a preface by Mr. Oscar Browning. This volume contains the essence of Herbart's pedagogy, and I have freely drawn upon it for the purposes of this note.]

W. J. Greenstreet.

MATHEMATICAL WORTHIES.

I. Edward Wright.

Edward Wright was probably born about 1560, and he died in 1615. He was a fellow of Caius College, Cambridge. The chief source of information about him, besides his own published works, is an obituary notice in a Latin paper preserved in the college library. He furnishes a notable instance of the conditions under which mathematical science was pursued in his time. The main object kept in view was to supply material needs of life, principally at this period in all matters connected with navigation, and the promoters and supporters of research in this direction were not the universities, but the trading and mercantile communities. Wright was appointed by the East India Company their Lecturer on Mathematics, at a salary of £50 a year. He was mechanical tutor to the eldest son of James I., Prince Henry, who died in 1612 at the age of nineteen. For his pupil's instruction he designed and made a wonderful piece of mechanism, displaying all the movements of the heavenly bodies

then known, and constructed to indicate these correctly for 17,100 years. It lasted, in fact, but a very few years. During the troubles of the civil war it was cast away as old rubbish, and in 1646 was found by Sir Jonas Moore, and deposited in its dilapidated state at his house in the Tower.

From the time of the discovery of Algoristic arithmetic the pathway of practical science was northward across the Continent to England, and thus our early mathematicians were constantly on the watch to see what was going on in Italy and the Netherlands. Wright gave us a translation of a work of Stevin's called *The Haven-finding Art,* and he was the first to explain the principle upon which depended Mercator's projection, the inventor, Kauffman, having applied merely rule-of-thumb processes. In a historical dissertation prefixed to Robertson's *Navigation,* it is stated that nearly all that was done to advance navigation in the seventeenth and eighteenth centuries was the work of three men—Wright, Norwood, and Halley. With Napier, the inventor of logarithms, are associated the names of Briggs and Wright, as assisting in the promotion and improvement of this means of calculation. Wright translated into English Napier's great work on the *Description of Logarithms.*[1]

There are some other matters of interest with which Wright had to do. He was the first to suggest the establishment of a standard length by the division of a meridian circle. The determination of the longitude was an unsolved problem in Wright's time. The idea of doing this by means of a clock had been suggested, but no clocks were accurate enough for the purpose. Wright elaborated a method which depended on the variation of the compass, sound and good in itself, but which improvements in clock-making rendered useless. The project of supplying London with water brought from springs near Ware, in Hertfordshire, was conceived and all but carried out by him. He was not, however, as able to finance schemes as to put them into shape for others to make use of. Wright was easily pushed aside, and Sir Hugh Middleton had all the glory of the famous work called the New River. No more fitting close to a notice of this too little-known mathematician can be given than the last few lines of the Latin paper above referred to: "Of him it may truly be said that he studied more to serve the public than himself, and though he was rich in fame and in the promises of the great, yet he died poor, to the scandal of an ungrateful age."

G. Heppel.

[1] Napier's other work on logarithms (*Mirifici Logarithmorum canonis constructio*) was first translated into English in 1889 by W. R. Macdonald.

NOTES ON "A. I. G. T. SYLLABUS OF ELEMENTARY DYNAMICS." Part II.

(Macmillan & Co.)

The following notes may be found useful to those who are using the Syllabus.

Addendum to § 24, *to follow line* 6.—If two of the forces are parallel, this condition only necessitates that the third shall be parallel to them. Its line of action must be determined some other way, such as that of § 26, or of § 31.

Addendum to § 31.—If the two given forces are parallel, the above construction fails. But the theorem of moments still holds.

For, let p, q (Fig. i.) be the given parallel forces. Replace p by two components r, s, and let t be the resultant of q, s, so that r, t are equivalent to p, q. Then, generally, r, t will not be parallel. And therefore, by this section, the sum of the moments of p, q about any point = the sum of the moments of r, s, q = the sum of the moments of r, t = the moment of the resultant.

If the forces p, q had been equal and opposite parallel forces (Fig. ii.), the forces r, t would also have been equal and opposite parallel forces, so that in this case the new method also fails. For this case see §§ 33, 34.

DIRECTOR CIRCLE OF A CONIC INSCRIBED IN A TRIANGLE.

1. Let TP, TQ (Fig. i.) be tangents to a conic, C its centre, S, H its foci, $2a$ and $2b$ its axes. From S draw a perpendicular to TP and produce it to its image S′; then we know that S′T = ST, angle S′TP = PTS, and S′H = major-axis = $2a$. Similarly with a perpendicular drawn from H to TQ and produced to its image H′. Thus the two triangles S′TH and STH′ are equal, and the angles S′TS, HTH′, and therefore the halves of these angles, are equal, that is, "the tangents from T are equally inclined to the focal distances of T."

Taking either of the above triangles, we have

$$4a^2 = ST^2 + TH^2 - 2ST \,.\, TH \cos PTQ$$
$$= 2CS^2 + 2CT^2 - 2ST \,.\, TH \cos T,$$
$$\therefore a^2 + b^2 = CT^2 - ST \,.\, TH \cos T.$$

2. Let the tangents TP. TQ be cut in K, L, by the tangent at a third point R, and in k, l by the tangent

parallel to it. Then the rectangle $kP \cdot PK$

$= $ sq. on radius parallel to PK

$= SP \cdot PH$. Also angles SPk, HPK are equal.

$\therefore$ the triangles SPk, HPK are similar

$$\therefore \text{angle } PkS = PHK = \tfrac{1}{2}PHR = \tfrac{1}{2}PHQ - \tfrac{1}{2}QHR = THQ - LHQ = THL;$$

and the angles kTS, HTL have been shown equal (1), therefore these triangles kTS, HTL are similar and the rectangle $kT \cdot TL = ST \cdot TH =$ constant.

3. The result of Article 1 may now be written

$$a^2 + b^2 = CT^2 - kT \cdot TL \cos T.$$

Now draw LF (Fig. ii.) at right angles to TK, and TD at right angles to KL, passing through the orthocentre O, and cutting kl in d. Also draw CV parallel to the tangents, and therefore bisecting Dd at right angles: and let us suppose the triangle TKL acute-angled, so that O falls within it.

The above result now becomes

$$\begin{aligned} a^2 + b^2 &= CT^2 - kT \cdot TF \\ &= CT^2 - TO \cdot Td \\ &= TO \cdot TD + CT^2 - TO(TD + Td) \\ &= TO \cdot OD + TO^2 + CT^2 - 2TO \cdot TV \\ &= TO \cdot OD + CO^2. \end{aligned}$$

[If, instead of TKL, we had taken the circumscribed triangle Tkl and o its orthocentre, we should have found in the same way $a^2 + b^2 = To \cdot od + Co^2$.]

Now make O the centre of a circle with radius $= \sqrt{TO \cdot OD}$, and the conclusion runs thus: "The director circle of any conic touching the sides of an acute-angled triangle cuts this circle at the extremities of a diameter."

If the triangle be obtuse-angled, so that O falls outside it (in which case the above circle is called the polar circle of the triangle), a similar investigation leads to the result $a^2 + b^2 = -TO \cdot OD + CO^2$, and the interpretation is, "The director circle of any conic touching the sides of an obtuse-angled triangle cuts the polar circle of the triangle orthogonally."

4. Hence follow easy proofs of some well-known propositions.

α. If the conic be a parabola, the director circle becomes the directrix, and we have, "The directrix passes through the orthocentre."

β. If the conic be an in-circle or ex-circle and r its radius, $a^2 + b^2$ becomes $2r^2$, and we have the property which has been used to prove Feuerbach's theorem that the nine-point circle touches each of the in- and ex-circles (see for example Richardson and Ramsey's *Modern Plane Geometry*, pp. 32, 34).

From this property we can deduce a solution of a problem set in a scholarship paper of King's College, Cambridge, for December 1892. "Given C the centre of the inscribed circle of a triangle, and P its point of contact with a side, and O the orthocentre, construct the triangle." Through P draw a perpendicular to PC, and from O draw OD at right angles to it. With centre C and radius $= CP\sqrt{2}$ describe a circle cutting OC produced both ways in G . G′. The point where a circle through GDG′ cuts OD again is a vertex of the triangle, and the rest follows at once.

γ. Let a conic touch the sides of a quadrilateral. Complete the quadrilateral. The above investigation applies to the orthocentre of each of the four triangles formed by the tangents, and to each of the conics touched by them; and as one of these conics can be a parabola we have this result: "The orthocentres of the four triangles of a complete quadrilateral lie on one straight line," viz. the directrix of the parabola.

δ. Except in the case of right-angled triangles, which we may neglect as being easy, we can see from a figure that two at least of these four triangles are obtuse-angled. Let us confine our attention to two obtuse-angled triangles and the corresponding polar circles. As the director circles of all inscribed conics cut these polar circles orthogonally, it follows that "the centres of the director circles, and, therefore, of the conics, lie on a straight line." This line is the radical axis of the polar circles, and is therefore perpendicular to the line of orthocentres. Also, it is easily seen that "each of the director circles cuts the line of orthocentres at two fixed points."

ϵ. Each diagonal of a complete quadrilateral is the ultimate form of an inscribed ellipse. Hence "The middle points of the three diagonals lie on a straight line," and "The centre of an inscribed conic lies on the straight line joining the middle points of the diagonals."

ζ. Lastly, the director circle corresponding to each of these diagonals is the circle which has that diagonal for its diameter, for the expression $a^2 + b^2$ becomes $a^2 + O$. Hence "The circles on the three diagonals of a complete quadrilateral pass through the same two points," viz. the two fixed points on the line of orthocentres.

I reserve for another paper the propositions which can be deduced from a special case of Article 2.

E. P. Rouse.

SOLUTIONS OF EXAMINATION QUESTIONS.

The Editor will be glad to avail himself of the help of all classes of readers towards making this section of the Gazette *as useful as possible.* Mathematical Tutors *are invited to send neat solutions;* Students *to call attention to classes of problems presenting exceptional difficulties, and* Examiners *who sympathise with us to forward copies of their papers. The help of foreign readers is especially requested in obtaining copies of papers set in the public examinations of other countries.*

7. *A string hangs vertically from one side of a horizontal circular cylinder of given radius, and carries a heavy particle at its lower end. Find the least velocity with which the particle must be projected horizontally in order that the string may wrap itself round the cylinder, never becoming slack.*

[Inter. Arts Hons. 92.]

Assume string a longer than $\frac{3}{2}\pi r$, so that the particle may at least reach the highest point of the cylinder and not slide down again after reaching it. Let B be the position of the particle when the string is vertically upright. The tension here may be put zero. This will happen if at B $mv^2/(a-\pi r)=mg$.

But, if V is the initial horizontal velocity, the equation of energy gives—

$$\tfrac{1}{2}mV^2=\tfrac{1}{2}mv^2+mg(2a-\pi r),$$

where v is the velocity at B;

$$\therefore\ V^2=g(a-\pi r)+2g(2a-\pi r),$$
$$=g(5a-3\pi r).$$

Note (i) that the centre of curvature of the particle is always the point where the string leaves the cylinder.

(ii) That if the particle gets round the first time it will *a fortiori* every future time permitted by the length of the string, since the actual velocity will by the energy equation be greater at each succeeding highest point, while the required minimum velocity will be less each time.

8. (i) *In a triangle* ABO *join* O *to any point* D *in* AB; *let* R *and* R_1 *be the radii of the circumcircles of triangles* AOD, BOD. *Show that* R . OB = R_1 . OA, *and that the distance between the centres* = R . AB ÷ OA.

(ii) *On the same chord and on the same side of it two arcs are described, each greater than a semicircle. Show how to draw through one end of the chord a line cutting both arcs, and such that its segment between them is the greatest possible.*

(iii) *Two unequal circles intersect at* D; *show how to draw a straight line* ADB, *cutting the one circle in* A *and the other in* B, *such that* AD = DB.

(iv) *If two equal circles are given intersecting in* O *and* D, *and if any straight line be drawn to meet the circles in* A *and* B, *find the locus of the mid point of* AB.

(v) *Let two circles intersect in* O *and* D. *In* OD *produced find a point* P *such that if tangents* PA, PB *be drawn to touch the circles respectively in* A *and* B, *the join of* AB *may pass through* D.

These exercises, selected from various Third Stage papers of the Science and Art Department, all illustrate an important general theorem on two intersecting circles.

(i) Find E and F, the centres of circles AOD, BOD. Join OE, EF, FO. Then EF bisects OD at right angles.

∴ angle OEF = $\frac{1}{2}$ angle OED
= angle OAD.
So angle OFE = angle OBD,
∴ triangles OEF, OAB are equiangular,
∴ OE : OF : : OA : OB,
∴ R . OB = R_1 . OA.
Also FE : EO : : BA : AO.
∴ EF = R_1 . AB ÷ OA.

We have incidentally demonstrated the following general theorem:—

(α) The triangle formed by any straight line (AB) drawn through one of the common points (D) of two given intersecting circles (AOD, BOD), and the two straight lines (OA, OB) drawn from the other one (O) to the points where it cuts the circles again, is of invariable form, and the constant ratio of these two straight lines is that of the radii of the circles.

Note that if the circles are equal the triangle is isosceles.

In the demonstration given A and B have been supposed on opposite sides of D. A slight modification only is required for other cases.

(ii) Since OAB is of invariable form, AB will be greatest when OB is greatest, that is, when it is a diameter of circle OBD.

(iii) Bisect EF in K. Make angle DOA equal to angle EOK.

∴ angle OAB = angle OEF,
∴ triangles OAD, OEK are equiangular,
∴ OA : AD : : OE : EK.
But AB : AO : : FE : EO,
∴ AB : AD : : FE : EK
∴ D is mid point of AB.

(iv) When the circles are equal OA = OB. Hence the line joining O to the mid point of AB is at right angles to AB. Hence the locus of the mid point of AB is the circle on OD as diameter.

(v) At O make angle DOA = $\frac{1}{2}$ angle EOF, and let the tangents at A and B meet at P.

Then angle AOD = angle BOD (by i),
∴ angle PAB = angle PBA.
∴ PA = PB.
∴ P lies on OD produced.

Theorem (α) is connected with the general theory of similar figures, and may be thus enunciated:—

Let two circles whose centres are E *and* F *intersect at a point* O. *Then the first can be transformed into the second by a "turn"* (= *angle* EOF) *round* O, *followed by a "stretch"* (= OF/OE) *from* O.

The point O is in fact a centre of "stretch-rotation" or "pivot-point" for the two circles. The student is referred to Petersen's *Methods and Theories* for a simple treatment of the theory, and an excellent set of illustrative examples.

Theorem (α) is also curiously connected with the geometry of the parabola and with some lemmas of Newton's *Principia* (Bk. I. § v. 23, 26, 27).

9. *In a given rhombus to inscribe the least possible rhombus.*
[Sc. and Art. 3rd Stage.]

Let the diagonals AC, BD of the given rhombus cross at E. Then those of any inscribed rhombus must also cross at E. We may, therefore, confine our attention to one of the four equal right-angled triangles, PEQ, into which its diagonals divide it.

If PEQ, P′EQ′ be two positions of such a triangle in which it has the same area,

PE . EQ = P′E . EQ′
∴ PE : P′E : : EQ′ : EQ.

But angle PEP′ = angle QEQ′,
∴ angle EPP′ = angle EQ′Q (vi. 6),
∴ angle BEP = angle BEQ′
= angle AEP′.

Hence by the principle of "*coincidence of equal values*," the only "turning value" for the area of triangle PEQ between the positions AEB, BEC is that in which EP bisects angle AEB.

The above investigation applies to the following much more general theorem:—

If a straight line PQ *cuts two fixed straight lines* AB, AC *in* P, Q, *and subtends a constant angle* PEQ *at a fixed point* E *within the angle* BAC, *the area of the triangle* PEQ *can only have a "turning value" when* EP, EQ *are equally inclined to* AB, AC.

We leave it as an exercise to the student to find out whether such a position always exists, and whether it is a position of maximum or minimum.

The following theorem can also be established by the same reasoning:—

If a chord PQ *of any given oval curve subtend a constant angle at a fixed point* E *within the oval, the positions in which the area of the triangle* PEQ *is a maximum or minimum are those in which* EP, EQ *are equally inclined to the tangents at* P *and* Q—

It is perhaps worth noticing that as the above conclusions were based—

(i) on the equality of triangles PEQ, P′EQ′,
(ii) „ „ angles PEQ, P′EQ′,

they would follow equally well if *angle* PEQ *was to be a maximum or minimum*, subject to the condition that triangle PEQ remained constant in area.

For instance, if EP, EQ were semiconjugates of a given ellipse, if the angle PEQ is to be a maximum or a minimum we must have EP, EQ equally inclined to the tangents at P and Q, that is EP, EQ must be equal—a well-known result, and leading obviously to the solution of the following:—

Determine the points on an ellipse at which any given diameter subtends the least angle, and show that the minimum angle is the same for all diameters.

[J., C., and E. Schol. 92.]

[For supplemental chords are parallel to conjugate diameters.]

10. *An open box in the shape of a perfect cube lies on a horizontal plane with the diagonals of its base pointing towards the four cardinal points. Show that at noon when the sun's altitude is tan* $^{-1}2\sqrt{2}$ *the portions of the interior surface in sunlight and in shade are as* 37 : 43. [W. 91.]

Let ABCD be the base of the box, AC the diagonal running north and south, A its southern end, AF the vertical edge common to the two faces on which the sun shines externally at noon, K the shadow of F, E the intersection of the diagonals. Then since tan AKF = $2\sqrt{2}$ therefore AK = $\frac{1}{2}$AE = $\frac{1}{4}$AC. In the diagram two sides of the box have been supposed removed. The shadows of the horizontal edges meeting in F (only one of which FG is shown) will be two straight lines KL, KM parallel to AB, AD. Hence the shadow of the side AFGB will be bounded on the face GC by the straight line GL. Take AB = 4; then BL = 1. Then area of GBL = 2.

Area in shadow = 2AFGB + 2GBL + ABLKMD
= 32 + 4 + (16 − 9)
= 43.
Area illuminated = 80 − 43
= 37.

11. *Straight lines all passing through a fixed point cut two fixed straight lines which are at right angles. Show that the locus of the mid point of the intercept between the fixed lines is a hyperbola: find its asymptotes and draw the curve.*

The question, though proposed for analytical treatment, yields readily to geometrical investigation. Let OX, OY be the two fixed straight lines (which need not be at right angles); P the fixed point; AB a straight line through P meeting OX, OY in A and B; Q the mid point of AB. Through the mid point C of OP draw CH, CK parallel to OX, OY, and meeting AB in HK. Then

HK = $\frac{1}{2}$AB = BQ,
∴ HQ = PK,

∴ Q lies on a hyperbola through P, having CH, CK for asymptotes.

The property used in the demonstration will enable us to find readily any number of points on this hyperbola, or on any other of which the asymptotes and one point are known. [See the Eighteenth Report of the *A. I. G. T.*, 1892 (*Geometrical Odds and Ends*). For a generalisation see Milne's *Companion to Problem Papers*, p. 255.]

A second solution is worthy of notice. With the construction shown in Fig. ii., RE . RF = PL . PM, and is therefore constant, therefore locus of R is a rectangular hyperbola with centre at P and asymptotes parallel to OX, OY.

But OQ = $\frac{1}{2}$OR, therefore locus of Q is a rectangular hyperbola with centre at mid point of OP and asymptotes parallel to OX, OY.

QUESTIONS FOR SOLUTION BY STUDENTS.

(18-21, PROF. NEUBERG. 22, 23, W. GALLATLY, M.A. 24-28, EDITOR.)

18. Soient OA, OB, OC trois droites situées dans un même plan. Par un point fixe M de OC on mène une transversale quelconque qui rencontre OA en E, OB en F, et l'on joint les points E, F à un second point fixe, N, de OC. Démontrez que la différence cot ENO − cot FNO est constante.

19. Décrire une circonférence qui passe par un point donné A qui soit vue d'un point donné B sous l'angle donné β et d'un point donné C sous l'angle donné γ.

20. On donne un triangle ABC. Par les sommets B, C

on mène deux droites parallèles qui rencontrent les côtés opposés AC, AB aux points E, F. Démontrez que la droite EF est tangente à une hyperbole fixe lorsque la direction des parallèles change.

21. Sur la base BC d'un triangle donné ABC on construit un triangle isoscèle quelconque BCD. Les côtés BD, CD de ce triangle rencontrent AC, AB respectivement en E et F. Démontrez que la droite EF enveloppe une conique inscrite au triangle ABC.

22. If R is the resultant of forces P and Q, S of forces P and R, and T of forces Q and R, prove that $S^2 + T^2 = P^2 + Q^2 + 4R^2$.

23. In a spherical triangle $\cos a = \frac{3}{5}$; $\cos b = \frac{5}{13}$; $C = 60°$. Show that $a = c$.

24. If two sides of a triangle are given in position and its perimeter in magnitude, show that the third side touches a fixed circle, without using the centre of the circle or the equality of tangents from the same point.

25. Give a statical proof that

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8}, \text{ etc. } ad\ inf. = \tfrac{1}{3}.$$

26. Illustrate by a single diagram that triangles whose sides are proportional to any one of the following sets of numbers have either an exterior or an interior angle of 60°—

(3, 5, 7), (5, 7, 8), (7, 8, 13), (7, 13, 15), (8, 13, 15).

27. Given $\log 1{\cdot}3712 = {\cdot}1371008$, $\mu = {\cdot}4343$, find x to 6 significant figures, where $10 \log x = x$.

28. Find a point P in the base BC of a triangle ABC at which the difference between the square on AP and the rectangle BP, PC is a maximum or minimum, and discuss the different cases that may arise.

[*The Editor would be glad to receive contributions to this column. Solutions of questions* 1, 2, 3, 4, 16 *by* "*Flamingo*" *have been received.*]

The Crafte of Nombrynge.

The constantly increasing importance that is being attached to the history of mathematics, shown not only by the continual succession of works specially devoted to it, but in the growing practice of appending historical notes to students' manuals, receives a further illustration in the announcement by the Early English Text Society of their intention to publish under the above title a fragment of one of the earliest treatises on Arithmetic in the language. Through the kindness of Dr. Furnivall proof copies of this work, extending to 30 pages, have been circulated among members of the A. I. G. T., and we take this opportunity of publicly tendering him our thanks. The volume of which it is to form part will also contain the Arithmetic by John of Halifax (about 1250), and Robert Recorde's treatise on the Counting Board. The editor (Mr. Robert Steele, of the Modern School, Bedford) would be glad to receive notice of any arithmetical MS., and of any information bearing on the subject of Medieval Arithmetic, which would serve to illustrate the passage from the Counting Board to computation on paper. Mr. Steele intends in his introduction to give illustrations of medieval methods, and of the development on the Counting Board of an arithmetic of position which he thinks independent of the Indian system. Believing with Mr. Heppel[1] in the advantage to the teacher of historical illustrations in mathematical instruction, and generally in the advantage to a craftsman of an acquaintance with the history of his craft, we wish the E. E. T. S. every success in its venture.

BOOKS AND MAGAZINES RECEIVED.

The Outlines of Quaternions. By LIEUT.-COL. H. W. L. HIME. Lieut.-Col. Hime neither quarrels with other writers nor proclaims an era of grotesque notations. The first part of his book deals with the addition and subtraction of vectors, and vector proofs are given of a large number of the propositions of Modern Geometry. Of the 150 pages forming the second part, 100 are devoted to the discussion of the multiplication and division of vectors, the quaternion proper, its various forms, its behaviour towards the laws of Algebra, the interpretation of results, and the differentiation of the quaternion. The remaining 50 pages deal with the application of quaternion analysis to pure Mathematics. Convenient tables of results are given. These are stated in words as well as symbols, a help to the beginner in reading quaternion expressions.

The present book will be welcome to those who wish to make an elementary acquaintance with Quaternion Analysis. Its style is simple and unpretentious, while it is a positive treat to read a Mathematical Text-Book which is not infested with examples from Examination Papers.

A. J. P.

Proceedings of the Edinburgh Mathematical Society. Vols. I.-XI.

Periodico di Matematica. March-April, June-August 1894.

Journal de Mathematiques Élémentaires. Nos. 1-6, 1894.

The Electrician, No. 834. (This contains an instalment of an important paper by Dr. J. Hopkinson, F.R.S., on "The Relation of Mathematics to Engineering," and a leading article on the comparative claims of graphical and analytical methods. Dr. Hopkinson's paper is concluded in No. 835.)

Indian Engineering, No. 22.

[1] See the Nineteenth Report (1893) of the A. I. G. T. (*The Use of History in Teaching Mathematics*).